ELSEVIER Journal of Pure and Applied Algebra 124 (1998) 241–259

JOURNAL OF PURE AND APPLIED ALGEBRA

Reconstruction in braided categories and a notion of commutative bialgebra

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Communicated by G.M. Kelly; received 17 August 1994; revised 25 May 1995

Abstract

A braided group in the sense of Majid [6] is a Hopf algebra B in a braided monoidal category which satisfies a generalized commutativity condition; this condition is expressed with respect to a certain class of B-comodules. The more obvious condition that B be a commutative algebra in the braided category does not make sense.

We propose a different commutativity condition for bialgebras: We show that a coalgebra reconstructed from a category over a braided base category $\mathscr{A}$ has the additional structure of being an object of the center $\mathscr{Z}(\mathscr{A}\text{-Coalg})$ of the category of coalgebras. We prove that braided groups which are reconstructed from braided monoidal categories over $\mathscr{A}$ are commutative algebras in the center of $\mathscr{A}$-Coalg. We give further information about Hopf algebras in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$.
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1991 Math. Subj. Class.: 18D10, 16W30, 16B50

1. Introduction

In noncommutative geometry of quantum groups and quantum spaces one often encounters noncommutative algebras whose noncommutativity is controlled by commutativity relations given in terms of solutions of the quantum Yang Baxter equation. As a unifying machine for noncommutative geometry of algebras with controlled noncommutativity Manin [7] has suggested the use of a symmetric base category within which all constructions should take place. We denote τ the symmetry of this category $\mathscr{C}$. The multiplication Δ of a commutative algebra in $\mathscr{C}$ then satisfies $\Delta \circ \tau = \Delta$. Note that underlying this "commutative" algebra there can still be a noncommutative ordinary algebra. A fruitful example of this setup is the category of $\mathbb{Z}_2$-graded vector spaces as a basis for supergeometry. If one wishes to extend the unifying machine of

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PII S0022-4049(96)00093-X

a base category with nonstandard commutativity of the tensor product to the case of quantum spaces given in terms of R-matrices, one has to drop the requirement that τ be a symmetry and deal with a braided base category $(\mathscr{A}, \tau)$ instead. Many notions of the theory of algebras and bialgebras still generalize nicely to braided categories – for example, the notions of a commutative algebra and of a bialgebra (Note the definition of a bialgebra involves a flip of tensor factors!) However, new problems arise from the fact that $\tau^2 \neq \mathrm{id}$.

The particular subtleties arising when one wants to deal with something like "commutative bialgebras" in a braided base category, were pointed out by Majid. He observed that if (B, ∇, Δ) is a bialgebra in a braided category $(\mathscr{A}, \tau)$, then $(B, \nabla\tau, \Delta)$ is not, as a rule, again a bialgebra. Thus we can rarely expect a bialgebra to be a commutative algebra in the sense that $\nabla = \nabla\tau$. Majid proposes to replace ordinary commutativity of a bialgebra B by a weaker condition, which has to be formulated with respect to a specified class of B-comodules. Thus a braided group in the sense of Majid is a bialgebra together with a class of comodules such that a compatibility condition is fulfilled for each of these comodules. A class of coalgebras that comes naturally equipped with a specified class of comodules consists of those obtained by reconstruction techniques from a category over $\mathscr{A}$, that is, from a category $\mathscr{C}$ and a functor $\omega : \mathscr{C} \to \mathscr{A}$. The coalgebra $C := \mathrm{coend}(\omega)$ reconstructed from $(\mathscr{C}, \omega)$ is the universal one such that ω factors through the underlying functor $\mathscr{M}^C \to \mathscr{A}$. (We omit here the technical assumptions that make such a reconstruction possible.) In sloppier language, C is the universal coalgebra having all the $\omega(X)$ as comodules. Now if $\mathscr{C}$ is a braided monoidal category and ω preserves the braiding, C is indeed a bialgebra, braided commutative with respect to the class $\{\omega(X) \mid X \in \mathscr{C}\}$ of comodules. Note, however, that to speak about this "commutativity" property of C, we have to keep track of the comodules $\omega(X)$ we started with when constructing C. In some sense this contrasts with the "philosophy" of reconstruction theorems, which strive to translate properties of a category $\mathscr{C}$ to properties of a (co)algebra of which the objects of $\mathscr{C}$ are (co)representations.

In this paper we will show that bialgebras that are commutative algebras with respect to a braiding can still be a useful concept in the study of braided groups, especially in the situation of bialgebras reconstructed from categories over a base category $(\mathscr{A}, \tau)$.

We have seen that $\nabla = \nabla\tau$ is not a reasonable condition on the multiplication ∇ of a bialgebra B in $\mathscr{A}$. However, if B is reconstructed from a braided category $\mathscr{C}$ and a braided functor $\omega : \mathscr{C} \to \mathscr{A}$, then we will find that $\nabla = \nabla\sigma$ holds for a new flip isomorphism $\sigma : B \otimes B \to B \otimes B$, which is both a solution of the QYBE and a coalgebra map. In fact, σ can be defined more generally, using the notion of the center of a monoidal category. We give a short review of the center construction in the preliminaries below; our references are Joyal and Street [1–3], and Kassel and Turaev [4]. The center $\mathscr{Z}(\mathscr{D})$ is a braided category that can be constructed starting from any monoidal category $\mathscr{D}$. If $\mathscr{D}$ is a braided category itself, then every object of $\mathscr{D}$ is an object of $\mathscr{Z}(\mathscr{D})$ in a standard way. Now assume $C = \mathrm{coend}(\omega)$ is a coalgebra reconstructed from a category $(\mathscr{C}, \omega)$ over $(\mathscr{A}, \tau)$. Then, while C is an object of $\mathscr{Z}(\mathscr{A})$ in the standard way, it does not become in this way an object of $\mathscr{Z}(\mathscr{A}\text{-Coalg})$, the center of the category of coalgebras in $\mathscr{A}$. In Proposition 5 we will find, however, that C is also an object of $\mathscr{Z}(\mathscr{A})$ in a new, natural but nonstandard way. This involves the definition of an isomorphism $\sigma_{\omega,X} : C \otimes X \to X \otimes C$ for each $X \in \mathscr{A}$. In Theorem 7 we will find that the new flip isomorphism has the property that $\sigma_{\omega,D}$ is a coalgebra morphism whenever D is a coalgebra. In this way, the reconstruction functor coend : $\mathfrak{C} \to \mathscr{A}\text{-Coalg}$ lifts to a braided functor $\mathrm{coend}_{\mathscr{Z}} : \mathfrak{C} \to \mathscr{Z}(\mathscr{A}\text{-Coalg})$. The result mentioned above now reads: *The bialgebra B reconstructed from a braided category $\mathscr{C}$ and a braided functor $\omega : \mathscr{C} \to \mathscr{A}$ is a commutative algebra in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$.*

The referee pointed out why one should expect such a functor exists. The functor coend : $\mathfrak{C} \to \mathscr{A}\text{-Coalg}$ has a right adjoint $\mathscr{M}^{-}$ that maps a coalgebra to the category of its comodules (we have omitted certain finiteness conditions we will discuss later). In case this right adjoint is fully faithful, the functor coend induces a braided monoidal functor $\mathscr{Z}(\mathrm{coend}) : \mathscr{Z}(\mathfrak{C}) \to \mathscr{Z}(\mathscr{A}\text{-Coalg})$, because in this case we have a natural isomorphism $C \cong \mathrm{coend}(\mathscr{M}^{C})$ for all coalgebras C (that means that the functor coend really "reconstructs" a coalgebra from its category of corepresentations, an ideal case that is known to occur when $\mathscr{A}$ is the category of vector spaces over a field). In this situation we can recover $\mathrm{coend}_{\mathscr{Z}} = \mathscr{Z}(\mathrm{coend}) \circ \mathscr{I} : \mathfrak{C} \to \mathscr{Z}(\mathfrak{C}) \to \mathscr{Z}(\mathscr{A}\text{-Coalg})$ where $\mathscr{I} : \mathfrak{C} \to \mathscr{Z}(\mathfrak{C})$ is the imbedding functor associated with the braiding of $\mathfrak{C}$.

Note that our commutativity result needs no reference to a collection of comodules for its statement. Instead the reconstruction has provided additional data on B. Although commutativity now takes the standard form $\nabla = \nabla\sigma$ again, this simplification is paid for, of course, with other subtleties: While B is a bialgebra with respect to the original braiding in $\mathscr{A}$, it is commutative with respect to a different braiding, namely that of $\mathscr{Z}(\mathscr{A}\text{-Coalg})$. However, the notion of a commutative algebra in the center of the category of coalgebras in $\mathscr{A}$ might be a good definition for some kind of braided affine semigroup within a braided base category.

There is another construction in which the new flip isomorphism can be used while the old one does not work. If B, H are two bialgebras in $(\mathscr{A}, \tau)$, then their tensor product algebra and tensor product coalgebra do not fulfill the compatibility condition for a bialgebra in $(\mathscr{A}, \tau)$. However, if B and H are bialgebras in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$, they have a natural tensor product algebra $B \otimes H$ in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$, which is in particular a bialgebra in $(\mathscr{A}, \tau)$. Note that the comultiplication of $B \otimes H$ is formed using the braiding τ of $\mathscr{A}$, while the multiplication uses the braiding of the center. We can show that if, in this situation B and H are Hopf algebras, then so is $B \otimes H$. The antipode is $S_{B\otimes H} = \sigma_{H,B}(S_H \otimes S_B)\tau_{B,H}$ and thus involves both the braiding of $\mathscr{A}$ and that of the center.

1.1. Preliminaries

Throughout the paper we will assume to be given a closed monoidal category $\mathscr{A}$ with braiding τ. That is, $\mathscr{A}$ has a tensor product bifunctor $\otimes$ which we assume to be strictly associative with unit object k, and $\tau : X \otimes Y \to Y \otimes X$ is a natural isomorphism satisfying $\tau_{X\otimes Y,Z} = (\tau_{X,Z} \otimes Y)(X \otimes \tau_{Y,Z})$ and $\tau_{X,Y\otimes Z} = (Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes Z)$, but not necessarily $\tau_{Y,X}\tau_{X,Y} = \mathrm{id}_{X,Y}$. We will assume that $\mathscr{A}$ has colimits and $\otimes$ preserves them in both arguments. We call an object X of $\mathscr{A}$ rigid if it has a left dual, that is, if there is X^* in $\mathscr{A}$ and morphisms $\mathrm{ev} : X^* \otimes X \to k$, $\mathrm{db} : k \to X \otimes X^*$ such that

$$X \xrightarrow{\mathrm{db}\otimes X} X \otimes X^* \otimes X \xrightarrow{X\otimes \mathrm{ev}} X$$
$$X^* \xrightarrow{X^*\otimes \mathrm{db}} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}\otimes X^*} X^*$$

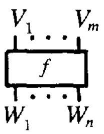
are the identity morphisms. Let $\mathscr{A}_0$ denote the full subcategory of rigid objects of $\mathscr{A}$. A category over $\mathscr{A}$ is a pair $(\mathscr{C},\omega)$, where $\mathscr{C}$ is a category and $\omega : \mathscr{C} \to \mathscr{A}_0$ is a functor. We refer to [8] for the following sketch of reconstruction theorems for bialgebras in $\mathscr{A}$. There are at least two useful definitions of a category of categories over $\mathscr{A}$. Both, $\mathfrak{C}_0$ and $\mathfrak{C}$, have objects the categories over $\mathscr{A}$. Morphisms $\mathscr{F} : (\mathscr{C},\omega) \to (\mathscr{D},\nu)$ in $\mathfrak{C}_0$ are functors $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ with $\nu\mathscr{F} = \omega$. A morphism $[\mathscr{F},\xi] : (\mathscr{C},\omega) \to (\mathscr{D},\nu)$ in $\mathfrak{C}$ is an equivalence class of pairs $(\mathscr{F},\xi)$, where $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is a functor and $\xi : \omega \to \nu\mathscr{F}$ is an isomorphism. We will call such a pair a functor over $\mathscr{A}$. The relevant equivalence relation is defined by $(\mathscr{F},\xi) \sim (\mathscr{G},\theta)$ iff there is $\phi : \mathscr{F} \xrightarrow{\sim} \mathscr{G}$ with $\nu(\phi) \circ \xi = \theta$. This equivalence relation makes sure that coend is a left adjoint functor, see [8, Theorem 2.1.12]. The right adjoint maps a coalgebra C to the category $\mathscr{M}^C_{\mathscr{A}_0}$ of those C-comodules that are contained in $\mathscr{A}_0$ (equipped with the obvious underlying functor). The coendomorphism coalgebra $C := \mathrm{coend}(\mathscr{C},\omega) := \mathrm{coend}(\omega)$ of a category $(\mathscr{C},\omega)$ over $\mathscr{A}$ is a representing object for the functor $\mathscr{A} \ni M \mapsto \mathrm{Nat}(\omega, \omega \otimes M) \in \mathscr{S}et$. Thus C, which always exists under the above assumptions on $\mathscr{A}$, satisfies $\mathrm{Mor}_{\mathscr{A}}(C,M) \xrightarrow{\Psi^\omega} \mathrm{Nat}(\omega, \omega \otimes M)$ for all $M \in \mathscr{A}$. In more detail, there is a natural transformation $\delta^\omega : \omega \to \omega \otimes C$ such that Ψ^ω defined by $\Psi^\omega(f) = (\omega \otimes f)\delta^\omega$ is bijective, that is, for each natural transformation $\phi : \omega \to \omega \otimes M$ there is a unique $f : C \to M$ with $\phi = (\omega \otimes f)\delta^\omega$. The category $\mathfrak{C}$ is a braided monoidal category, with $(\mathscr{C},\omega) \otimes (\mathscr{D},\nu) = (\mathscr{C} \times \mathscr{D}, \omega \otimes \nu)$, where $\omega \otimes \nu(X,Y) = \omega(X) \otimes \nu(Y)$, and with the braiding $[\mathscr{T},\tau] : (\mathscr{C},\omega) \otimes (\mathscr{D},\nu) \to (\mathscr{D},\nu) \otimes (\mathscr{C},\omega)$, where $\mathscr{T} : \mathscr{C} \times \mathscr{D} \to \mathscr{D} \times \mathscr{C}$ is transposition and $\tau : \omega(X) \otimes \nu(Y) \to \nu(Y) \otimes \omega(X)$ is the braiding in $\mathscr{A}$. The functor coend preserves tensor products: we have $\mathrm{coend}(\omega \otimes \nu) \cong \mathrm{coend}(\omega) \otimes \mathrm{coend}(\nu)$. The universal arrow is $\delta^{\omega\otimes\nu} = (\omega \otimes \tau \otimes \nu)(\delta^\omega \otimes \delta^\nu)$. Being a monoidal functor, coend maps algebras in $\mathfrak{C}$ to algebras in $\mathscr{A}$-Coalg. Thus, if $\mathscr{C}$ is a monoidal category and (ω, ξ) is a monoidal functor, then $C = \mathrm{coend}(\omega)$ is a bialgebra.

Furthermore, if $\mathscr{C}$ is rigid then C has an antipode.

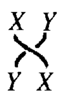
If the braiding τ is a symmetry then both, $(\mathscr{A}\text{-Coalg}, \tau)$ and $(\mathfrak{C}, [\mathscr{T}, \tau])$, are symmetric categories. In this case $\mathrm{coend} : \mathfrak{C} \to \mathscr{A}\text{-Coalg}$ is a symmetric functor, that means compatible with the symmetries involved.

The condition that $\mathscr{A}$ is closed means that the functors $\mathscr{A} \ni X \mapsto X \otimes Y \in \mathscr{A}$ have right adjoints $\underline{\mathrm{hom}}(Y,\text{-})$. Categories fulfilling all of these conditions are, for example, the category ${}_H\mathscr{M}$ of left modules over a quasitriangular k-Hopf algebra and the category $\mathscr{M}^H$ of right comodules over a coquasitriangular k-Hopf algebra which is k-flat. Note that here the underlying k-module of $\underline{\mathrm{hom}}(Y,X)$, which was constructed in [9], is not in general $\mathrm{Hom}_k(Y,X)$.

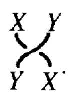
We will make constant use of the graphical calculus for doing computations in $\mathscr{A}$. References are Yetter et al. [11, 2]. [11, 3]. A morphism $f : V_1 \otimes \cdots \otimes V_m \to W_1 \otimes \cdots \otimes W_n$ will be symbolized by



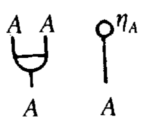
if no other graphical symbol is specified. Specifically we denote the braiding $\tau_{X,Y}$ by



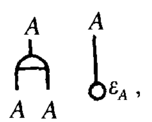
its inverse by



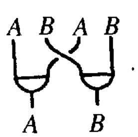
We will denote the multiplication and unit morphisms $\nabla : A \otimes A \to A$, $\eta : k \to A$ of an algebra and the comultiplication and counit morphisms $\Delta : C \to C \otimes C$, $\varepsilon : k \to C$ of a coalgebra by



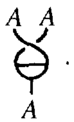
and



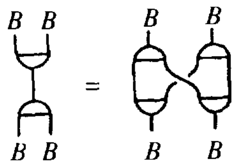
respectively. The tensor product of algebras A,B in $\mathscr{A}$ is $A \otimes B$ with the multiplication defined by



The tensor product of coalgebras is defined in the same way. The opposite algebra A^{op} of A is A with the multiplication



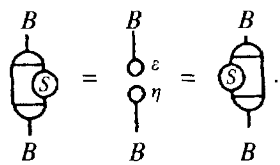
A bialgebra is an algebra in the monoidal category of coalgebras, or, equivalently, an algebra and coalgebra whose multiplication and unit are coalgebra maps; i.e.



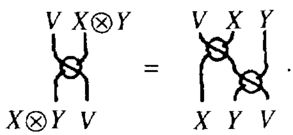
and other equations involving η and ε are required. We use

to denote a right C-comodule structure on $V \in \mathscr{A}$.

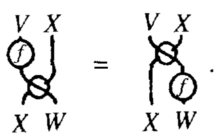
An antipode for the bialgebra B is a morphism $S : B \to B$ satisfying



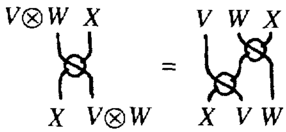
The center $\mathscr{Z}(\mathscr{D})$ of a monoidal category $\mathscr{D}$ is the category whose objects are pairs $(V, \sigma_{V,-})$ with $V \in \mathscr{D}$ and $\sigma_{V,X} : V \otimes X \to X \otimes V$ a natural isomorphism, depicted as , such that



A morphism in $\mathscr{Z}(\mathscr{D})$ is a morphism $f : V \to W$ which satisfies



$\mathscr{Z}(\mathscr{D})$ is a braided monoidal category with $(Y, \sigma_{V,-}) \otimes (W, \sigma_{W,-}) = (V \otimes W, \sigma_{V\otimes W,-})$, where $\sigma_{V\otimes W,X}$ is defined by



and the components of the braiding are the components of σ.

2. The braided reconstruction functor

Our aim is to provide a reconstructed coalgebra $C = \mathrm{coend}(\omega)$ with a new isomorphism $C \otimes D \to D \otimes C$ of coalgebras, defined for any coalgebra D and satisfying the braid relations (the precise statement, Theorem 7, involves the center construction). Let us show first that such an isomorphism arises very naturally provided that D is also a reconstructed coalgebra, say $D = \mathrm{coend}(\mathscr{D}, \nu)$. Recall that the category $\mathfrak{C}$ of all categories over $\mathscr{A}$ is braided monoidal with tensor product $(\mathscr{C}, \omega) \otimes_{\mathfrak{C}} (\mathscr{D}, \nu) = (\mathscr{C} \times \mathscr{D}, \omega \otimes \nu)$ and braiding $[\mathscr{T}, \tau]$. Put $\tilde{\tau}_{\omega,\nu} := \mathrm{coend}([\mathscr{T}_{\mathscr{C},\mathscr{D}}, \tau_{\omega,\nu}]) : C \otimes D \to D \otimes C$, where $(C, \Delta_C, \varepsilon_C) := \mathrm{coend}(\omega)$ and $(D, \Delta_D, \varepsilon_D) := \mathrm{coend}(\nu)$. Then

$$\tilde{\tau} = \mathrm{coend}(\mathscr{T}, \tau) : \mathrm{coend} \circ \otimes_{\mathfrak{C}} \to \mathrm{coend} \circ \otimes_{\mathfrak{C}}^{\mathrm{sym}}$$

is a natural morphism of $\mathscr{A}$-coalgebras. Since coend is a monoidal functor, we have

Lemma 1. *Let* $(\mathscr{F},\xi) : (\mathscr{C},\omega) \to (\mathscr{C}',\omega')$ *and* $(\mathscr{G},\vartheta) : (\mathscr{D},\nu) \to (\mathscr{D}',\nu')$ *be functors over* $\mathscr{A}$, *let* C, C', D *and* D' *be the corresponding coalgebras, and let* $f := \mathrm{coend}(\mathscr{F},\xi) : C \to C'$ *and* $g := \mathrm{coend}(\mathscr{G},\vartheta) : D \to D'$ *the corresponding coalgebra morphisms, respectively. We have*

(i) $(g \otimes f) \circ \tilde{\tau}_{\omega,\nu} = \tilde{\tau}_{\omega',\nu'} \circ (f \otimes g)$

(ii) $\tilde{\tau}_{\omega,\nu}$ *is a coalgebra isomorphism with* $\tilde{\tau}^{-1}_{\omega,\nu} = \mathrm{coend}(\mathscr{T}_{\mathscr{D},\mathscr{C}}, \tau^{-1}_{\omega,\nu})$.

(iii) *If* $(\mathscr{E}, \lambda)$ *is another category over* $\mathscr{A}$, *we have*:

$$\tilde{\tau}_{\omega,\nu\otimes\lambda} = (1 \otimes \tilde{\tau}_{\omega,\lambda})(\tilde{\tau}_{\omega,\nu} \otimes 1), \tag{1}$$

$$\tilde{\tau}_{\omega\otimes\nu,\lambda} = (\tilde{\tau}_{\omega,\lambda} \otimes 1)(1 \otimes \tilde{\tau}_{\nu,\lambda}). \tag{2}$$

If we assume that the right adjoint $\mathscr{M}^{-}_{\mathscr{A}_0}$ of coend is fully faithful, then Lemma 1 proves a lot more: Since every coalgebra is then in a functorial way a reconstructed one (the counits of the adjunction are isomorphisms $\mathrm{coend}(\mathscr{M}^{C}_{\mathscr{A}_0}) \cong C$), we find that the category $\mathscr{A}$-Coalg is braided. However, it is known that $\mathscr{M}^{-}_{\mathscr{A}_0}$ is not fully faithful even if $\mathscr{A} = \mathscr{M}^{H}$ for a nontrivial coquasitriangular Hopf algebra H. In fact, Majid's transmuted Hopf algebra construction, used as an example in Section 5 below, shows very explicitly that $k \in \mathcal{M}^H$-Coalg does not satisfy $k \cong \mathrm{coend}(\mathcal{M}^k_{\mathcal{A}_0})$. Still, we will be able to show that the isomorphism $\tilde{\tau}$ can be generalized dropping the assumption that D is a reconstructed coalgebra $D = \mathrm{coend}(\nu)$.

As a first step we will characterize $\tilde{\tau}_{\omega,\nu} = \tilde{\tau}_{\omega,D}$ without reference to ν.

Theorem 2. *$\tilde{\tau}_{\omega,\nu}$ is the unique morphism τ' that makes the following diagram commute:*

$$\begin{tikzcd}
\omega(X)\otimes D \arrow[r, "\delta^\omega\otimes 1"] \arrow[d, "\delta^\omega\otimes 1"] & \omega(X)\otimes C\otimes D \arrow[dd, "1\otimes\tau"] \\
\omega(X)\otimes C\otimes D \arrow[d, "1\otimes\tau'"] & \\
\omega(X)\otimes D\otimes C & \omega(X)\otimes D\otimes C \arrow[l, "\tau^2\otimes 1"']
\end{tikzcd}$$

Proof. By reconstruction theory, we know that $\tilde{\tau}$ is the unique morphism that makes the outer rectangle of the following diagram commute:

$$\begin{tikzcd}
\omega(X)\otimes\nu(Y) \arrow[r, "\delta\otimes\delta"] \arrow[dd, "\tau"] & \omega(X)\otimes C\otimes\nu(Y)\otimes D \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, phantom, "\mathrm{(I)}"] & \omega(X)\otimes\nu(Y)\otimes C\otimes D \arrow[d, "\tau\otimes\tau"] \arrow[dd, bend left=60, "\tau\otimes\tilde{\tau}"] \\
 & {} & \nu(Y)\otimes\omega(X)\otimes D\otimes C \ \mathrm{(II)} \arrow[d, "1\otimes\tau^2\otimes 1"] \\
\nu(Y)\otimes\omega(X) \arrow[r, "\delta\otimes\delta"] & \nu(Y)\otimes D\otimes\omega(X)\otimes C \arrow[r, "1\otimes\tau\otimes 1"] & \nu(Y)\otimes\omega(X)\otimes D\otimes C
\end{tikzcd} \cdot$$

(I) commutes since τ is a braiding. Since $\delta^\omega \in \mathrm{Nat}(\omega, \omega \otimes C)$ is not generally an epimorphism, (II) will not commute. But the two ways "first to the right then down" commute. These two ways can be easily, by using only the properties of $\otimes$ and τ, transformed to:

$$\begin{tikzcd}
\omega(X)\otimes\nu(Y) \arrow[r, "1\otimes\delta^\nu"] & \omega(X)\otimes\nu(Y)\otimes D \arrow[r, "\tau\otimes 1"] & \nu(Y)\otimes\omega(X)\otimes D \arrow[r, "1\otimes\delta^\omega\otimes 1"] \arrow[d, "1\otimes\delta^\omega\otimes 1"] & \nu(Y)\otimes\omega(X)\otimes C\otimes D \arrow[dd, "1\otimes 1\otimes\tau"] \\
 & & \nu(Y)\otimes\omega(X)\otimes C\otimes D \arrow[d, "1\otimes 1\otimes\tilde{\tau}"] & \\
 & & \nu(Y)\otimes\omega(X)\otimes D\otimes C & \nu(Y)\otimes\omega(X)\otimes D\otimes C \arrow[l, "1\otimes\tau^2\otimes 1"']
\end{tikzcd} .$$

Therefore $\tilde{\tau}$ is the unique morphism for which this diagram commutes. So if we have τ' as given in the theorem, we see that $\tau' = \tilde{\tau}$, since $\tilde{\tau}$ is unique. If, on the other hand, we start with $\tilde{\tau}$ we transform the last diagram to

$$\omega(X) \otimes v(Y) \xrightarrow{\tau} v(Y) \otimes \omega(X) \xrightarrow{\delta^v \otimes 1} v(Y) \otimes D \otimes \omega(X)$$

$$\xrightarrow{1 \otimes \tau^{-1}} v(Y) \otimes \omega(X) \otimes D \underset{1 \otimes g}{\overset{1 \otimes f}{\rightrightarrows}} v(Y) \otimes \omega(X) \otimes D \otimes C$$

Here we abbreviated $f := (\tau^2 \otimes 1)(1 \otimes \tau)(\delta^\omega \otimes 1)$ and $g := (1 \otimes \tilde{\tau})(\delta^\omega \otimes 1)$. Now the assertion follows by the universal property of δ^v, since $\omega(X)$ is rigid. □

Now we have characterized $\tilde{\tau}$ by a property which does not explicitly involve the functor v that D is reconstructed from, and we would like to make this property the defining property of a generalization of $\tilde{\tau}$ to arbitrary coalgebras D (while C still has to be a reconstructed coalgebra). To do this, we have to generalize the universal property of $C = \mathrm{coend}(\omega)$, utilizing (for the first time) our general assumption that $\mathscr{A}$ is closed.

Lemma 3. *Let $(\mathscr{C}, \omega) \in \mathfrak{C}$ and $C := \mathrm{coend}(\mathscr{C}, \omega)$. There is an isomorphism, natural in $N, M \in \mathscr{A}$,*

$$\Psi_{\omega,N,M} : \mathrm{Mor}(C \otimes N, M) \to \mathrm{Nat}(\omega \otimes N, \omega \otimes M)$$
$$f \mapsto (1 \otimes f)(\delta^\omega \otimes 1).$$

Proof. Since $\mathscr{A}$ is closed, there are natural isomorphisms for all $X, Y \in \mathscr{A}_0$:

$$\mathrm{Mor}(X \otimes N, Y \otimes M) \cong \mathrm{Mor}(Y^* \otimes X \otimes N, M) \cong \mathrm{Mor}(Y^* \otimes X, \underline{\mathrm{hom}}(N, M))$$
$$\cong \mathrm{Mor}(X, Y \otimes \underline{\mathrm{hom}}(N, M))$$

Since these isomorphisms are natural in all arguments, we have

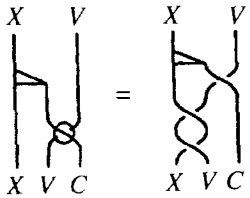
$$\mathrm{Nat}(\omega \otimes N, \omega \otimes M) \cong \mathrm{Nat}(\omega, \omega \otimes \underline{\mathrm{hom}}(N, M))$$
$$\cong \mathrm{Mor}(C, \underline{\mathrm{hom}}(N, M)) \cong \mathrm{Mor}(C \otimes N, M) \qquad \square$$

Using Lemma 3, we see that the morphism τ' from Theorem 2 is well defined not only for coendomorphism coalgebras D, but for arbitrary objects $V \in \mathscr{A}$.

Definition 4. Let $(\mathscr{C}, \omega) \in \mathfrak{C}$ and $C := \mathrm{coend}(\omega)$. For any $V \in \mathscr{A}$, define $\sigma_{\omega,V}$: $C \otimes V \to V \otimes C$ to be the unique morphism satisfying

$$\Psi_{\omega,V,V\otimes C}(\sigma_{\omega,V}) = (\tau_{V,\omega} \otimes C)(\tau_{\omega,V} \otimes C)(\omega \otimes \tau_{C,V})(\delta^\omega \otimes V).$$

In a pictorial representation this definition can be expressed by



where  stands for $\sigma_{\omega,V}$ and X is any element of $\mathscr{C}$. (For shortness we write X instead of $\omega(X)$, though everything takes place in $\mathscr{A}$.) Note that this is indeed a generalization, since for any $(\mathscr{D}, v) \in \mathfrak{C}$ with $D := \mathrm{coend}(v)$, the morphism $\tilde{\tau}_{\omega,v}$ from Theorem 2 agrees with $\sigma_{\omega,D}$.

The following proposition collects the coherence properties of σ. Note that while the properties of $\tilde{\tau}$ listed above were perfectly straightforward consequences of the braiding structure of $\mathfrak{C}$ and the fact that coend is a monoidal functor, we need entirely different techniques now, since the more general σ is not constructed as the image of a braiding under a functor.

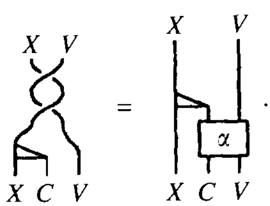
Proposition 5. *The mapping* $(\mathscr{C}, \omega) \mapsto (\mathrm{coend}(\omega), \sigma_{\omega,-})$ *defines a braided monoidal functor* $\widehat{\mathrm{coend}} : \mathfrak{C} \to \mathscr{Z}(\mathscr{A})$.

Proof. To show that $(\mathrm{coend}(\omega), \sigma_{\omega,-})$ is in the center of $\mathscr{A}$ we have to check that $\sigma_{\omega,V} : C \otimes V \to V \otimes C$ is a natural isomorphism and satisfies the coherence conditions

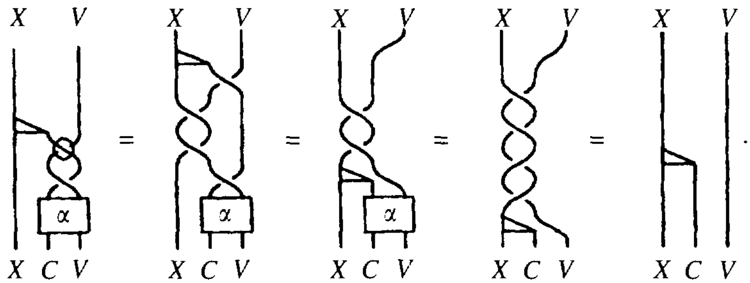
$$\sigma_{\omega,I} = \mathrm{id}, \tag{3}$$

$$\sigma_{\omega,V\otimes W} = (1 \otimes \sigma_{\omega,W})(\sigma_{\omega,V} \otimes 1). \tag{4}$$

The verifications are straightforward. We only show that $\sigma_{\omega,V}$ is invertible for all objects $V \in \mathscr{A}$. To do this, we define a morphism $\alpha : C \otimes V \to C \otimes V$ by $\Psi_{\omega,V,C\otimes V}(\alpha) = (\delta^{\omega} \otimes V) \circ (\tau_{V,\omega}^{-1} \circ \tau_{\omega,V}^{-1})$. In the pictorial calculus that means



We claim that σ is an isomorphism with $\sigma^{-1} = \alpha \circ \tau$. Thus we have to prove $(\alpha\circ\tau)\circ\sigma = \mathrm{id}_{C\otimes V}$ and $\sigma \circ (\alpha \circ \tau) = \mathrm{id}_{V\otimes C}$. Lemma 3 shows us that it is sufficient to prove this equation after an application of $\Psi_{\omega,V,W\otimes C}$. Let $X \in \mathscr{C}$ and $V \in \mathscr{A}$.



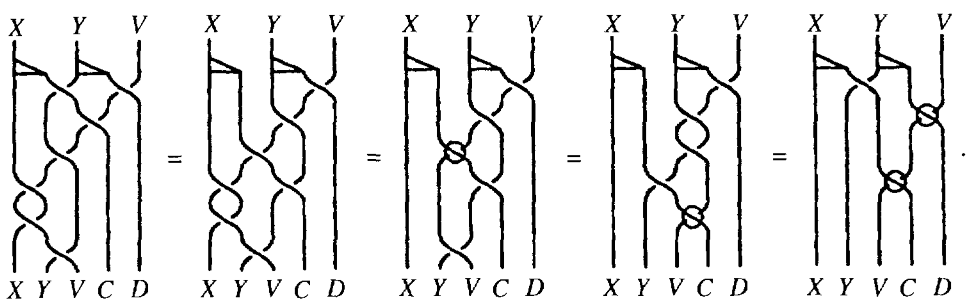
The second equation is equivalent to $\sigma \circ \alpha = \tau^{-1}_{C\otimes V}$. This can be verified similarly to the first equation.

By now we know that $(\mathrm{coend}(\omega), \sigma_{\omega,-})$ lies in the center of $\mathscr{A}$. To show that $\widehat{\mathrm{coend}}$ is a functor we have to check that $\sigma_{\omega(-),V}$ is natural in its first argument, i.e. $\sigma_{\nu,V} \circ (\mathrm{coend}(\mathscr{F})\otimes 1) = (\mathrm{coend}(\mathscr{F})\otimes 1)\circ\sigma_{\omega,V}$ for all functors $(\mathscr{F},\xi) : (\mathscr{C},\omega) \to (\mathscr{D},\nu)$. This can be verified easily using the definition of $\mathrm{coend}(\mathscr{F})$.

It remains to show that $\widehat{\mathrm{coend}}$ is monoidal and braided. For the first we have to prove a counterpart to Eq. (2) for all $(\mathscr{C},\omega), (\mathscr{D},\nu) \in \mathfrak{C}$, namely

$$\sigma_{\omega\otimes\nu,V} = (\sigma_{\omega,V} \otimes \nu)(\omega \otimes \sigma_{\nu,V}).$$

Let $X \in \mathscr{C}$ and $Y \in \mathscr{D}$. The first picture in the following diagram represents the left hand side of this equation:



That the functor $\widehat{\mathrm{coend}}$ is braided is obvious from the original definition of $\tilde{\tau}$ as the image of the braiding $(\mathscr{T},\tau)$ in $\mathfrak{C}$ under coend. $\square$

We have shown that the underlying object in $\mathscr{A}$ of the coalgebra $C = \mathrm{coend}(\omega)$ is an object of $\mathscr{Z}(\mathscr{A})$ in a natural, but nonstandard way. Of course any object of $\mathscr{A}$ lies in $\mathscr{Z}(\mathscr{A})$ since $\mathscr{A}$ is braided. The key advantage of the new structure $\sigma_{\omega,-}$ is that it is also compatible with the comultiplication Δ of C.

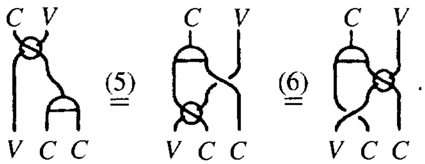
Lemma 6. *Let* $(\mathscr{C},\omega) \in \mathfrak{C}$ *with* $(C,\Delta_C,\varepsilon_C) := \mathrm{coend}(\omega)$ *and* $V \in \mathscr{A}$. *The morphism* $\sigma_{\omega,V} : C \otimes V \to V \otimes C$ *fulfills the following two compatibility relations with the coalgebra structure morphism* Δ_C:

$$(1 \otimes \Delta_C)\sigma_{\omega,V} = (\sigma_{\omega,V} \otimes C)(C \otimes \tau_{C,V})(\Delta_C \otimes V), \tag{5}$$

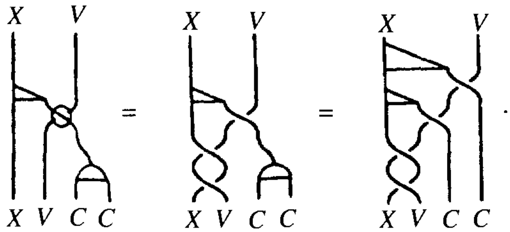
$$(1 \otimes \Delta_C)\sigma_{\omega,V} = (\tau_{C,V}^{-1} \otimes C)(C \otimes \sigma_{\omega,V})(\Delta_C \otimes V). \tag{6}$$

Moreover, for the morphism ε_C, *we have* $\Psi_{\omega,V,V}((1 \otimes \varepsilon_C)(\sigma_{\omega,V})) = \tau_{V,\omega}\tau_{\omega,V}$, *where* Ψ *is the isomorphism from Lemma* 3.

In the graphical calculus, these two commutation relations are represented by

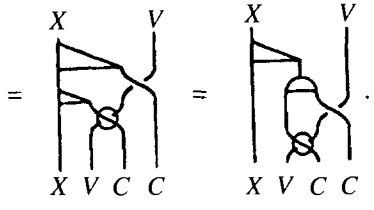


Proof. We start the computation with the following transformation:

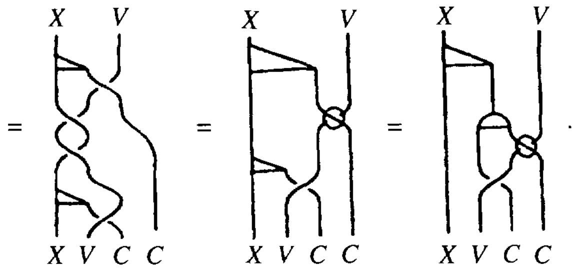


From this point the proof for the two equations splits.

To prove Eq. (5), we have to carry on our transformation in the following way:



The second Eq. (6) results from



We omit the easy proof for the assertion on ε_C. □

Note that Lemma 6 shows that Δ_C is not a morphism in $\mathscr{Z}(\mathscr{A})$, unless $\tau_{-,\omega}\tau_{\omega,-} = \mathrm{id}_{\omega,-}$. Hence coend($\omega$) is not in general an object in $\mathscr{Z}(\mathscr{A})$-Coalg. Do not confuse this fact with the statement that coend(ω) is an object in $\mathscr{Z}(\mathscr{A}$-Coalg), which we are going to prove now.

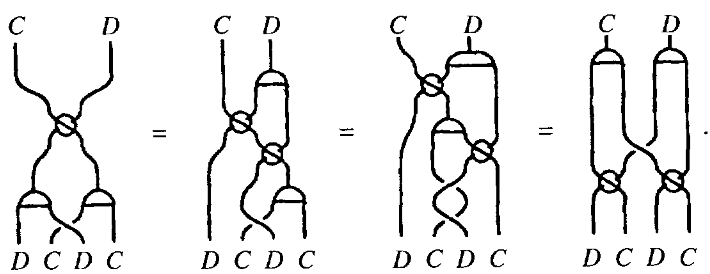
Theorem 7. *The mapping* $(\mathscr{C},\omega) \mapsto (\mathrm{coend}(\omega), \sigma_{\omega,-})$ *defines a braided monoidal functor* $\mathrm{coend}_{\mathscr{Z}} : \mathfrak{C} \to \mathscr{Z}(\mathscr{A}\text{-Coalg})$. *Moreover* $\mathrm{coend}_{\mathscr{Z}}$ *is left adjoint to the functor* $\mathscr{Z}(\mathscr{A}\text{-Coalg}) \ni (C,\sigma) \mapsto \mathscr{M}^C_\sigma$, *where* $\mathscr{M}^C_\sigma$ *is the category of all C-comodules for which the diagram of Theorem 2 with τ' replaced by σ commutes for each coalgebra D.*

Proof. All that is left to do to prove the first assertion is to show that for an arbitrary coalgebra $(D, \Delta_D, \varepsilon_D) \in \mathscr{A}$-Coalg the morphism $\sigma_{\omega,D} : C \otimes D \to D \otimes C$ is a coalgebra morphism. To prove this claim we observe that, since $\sigma_{\omega,D}$ is natural in D, the morphism $\sigma_{\omega,D}$ commutes with the coalgebra structure maps of D.

$$(\Delta_D \otimes C)\sigma_{\omega,D} = \sigma_{\omega,D\otimes D}(C \otimes \Delta_D), \tag{7}$$

$$(\varepsilon_D \otimes C)\sigma_{\omega,D} = C \otimes \varepsilon_D. \tag{8}$$

The following transformation yields the result:



The first equality is an application of Eq. (7), the second and third are applications of Eqs. (6) and (5), respectively.

We omit the easy proof that $\sigma_{\omega,D}$ is also compatible with the counit $\varepsilon_C \otimes \varepsilon_D$. The second claim is deduced easily from Theorem 2.1.12 in [8]. □

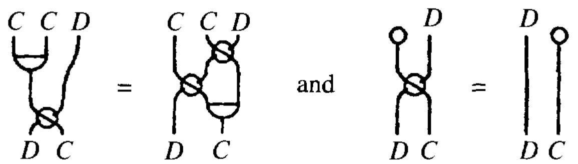
3. Commutative bialgebras

It is well known that if $(\mathscr{C},\otimes)$ is a monoidal category and ω a monoidal functor, then $C := \mathrm{coend}(\omega)$ is a bialgebra, that is, an algebra in the category of coalgebras in $\mathscr{A}$. Actually Theorem 7 allows one to show more:

Theorem 8. *Let $(\mathscr{C},\otimes,\omega)$ be a monoidal category over $\mathscr{A}$. Then $C := \mathrm{coend}_{\mathscr{Z}}(\omega)$ is an algebra in $\mathscr{Z}(\mathscr{A}$-Coalg).*

Proof. $\mathrm{coend}_{\mathscr{Z}}$, being a monoidal functor, maps monoidal categories over $\mathscr{A}$, that is, algebras in $\mathfrak{C}$, to algebras in $\mathscr{Z}(\mathscr{A}$-Coalg). □

Note that the theorem states that $\nabla : C \otimes C \to C$ and $\eta : k \to C$ are morphisms in $\mathscr{Z}(\mathscr{A}$-Coalg), that is



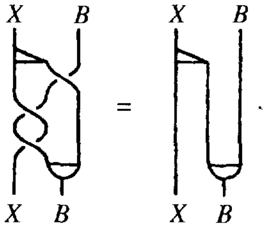
for all coalgebras D. If (B,Δ,∇) is a bialgebra in $\mathscr{A}$, then $(B,\Delta,\nabla\tau)$, that is, B with the same coalgebra and opposite algebra structure is not in general a bialgebra. Thus there is no good notion of opposite bialgebra in $\mathscr{A}$. Assume, however, that B is an algebra in $\mathscr{Z}(\mathscr{A}$-Coalg). Then, since $\mathscr{Z}(\mathscr{A}$-Coalg) is braided, there is a straightforward opposite algebra B^{op} in $\mathscr{Z}(\mathscr{A}$-Coalg), which, in particular, is a bialgebra. In more detail, the coalgebra structure of B^{op} is the same as that of B, while the algebra structure is

.

A braided monoidal category $(\mathscr{C},\otimes,\omega) \in \mathfrak{C}$ with a braided monoidal functor $\omega : \mathscr{C} \to \mathscr{A}_0$ is a commutative algebra in $\mathfrak{C}$. We know that $\mathrm{coend}_{\mathscr{Z}}$ is a braided monoidal functor between two braided monoidal categories, namely $\mathfrak{C}$ and $\mathscr{Z}(\mathscr{A}$-Coalg). Since these types of functors preserve commutative algebras we have

Theorem 9. *Let $(\mathscr{C}, \otimes, \omega)$ be a braided monoidal category with a braided monoidal functor $\omega : \mathscr{C} \to \mathscr{A}_0$. Then* $\mathrm{coend}_{\mathscr{Z}}(\omega)$ *is a commutative algebra in* $\mathscr{Z}(\mathscr{A}\text{-Coalg})$.

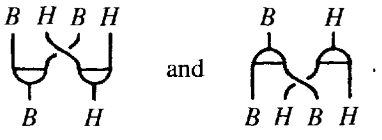
In [6] Majid has introduced a commutativity condition for bialgebras, which depends on a class of comodules: A bialgebra B in a braided category $\mathscr{A}$ is commutative with respect to a comodule $(X, \delta) \in \mathscr{M}^B$ if



Now suppose that B is a bialgebra whose underlying coalgebra is reconstructed from a category $(\mathscr{C}, \omega)$ over $\mathscr{A}$. Then the left hand side of this equation is equal to $(X \otimes \nabla\tilde{\tau})(\delta \otimes B)$, whenever X is in the image of ω. By Lemma 3, it follows that B is braided commutative in the sense of Majid with respect to all the comodules in the image of ω, iff B is commutative in the sense that $\nabla = \nabla\tilde{\tau}$; in particular, this always holds if B is the bialgebra reconstructed from a braided monoidal category $\mathscr{C}$ and a braided functor ω.

4. Tensor product bialgebras

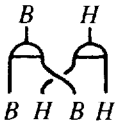
Assume B and H are bialgebras in $\mathscr{A}$. Then we can form their tensor product algebra and tensor product coalgebra both called $B \otimes H$ with multiplication and comultiplication, defined by



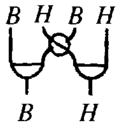
However with this construction $B \otimes H$ does not fulfill the compatibility condition for a bialgebra. The situation is no better if we replace one of the instances of τ in the construction with τ^{-1}. Indeed if $(\nabla_A \otimes \nabla_B)(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B)$ is a coalgebra morphism for some $\beta : A \otimes B \to B \otimes A$, then, since η_A and η_B are coalgebra morphisms, $\beta = (\nabla_A \otimes \nabla_B)(\eta_A \otimes \beta \otimes \eta_B)$ is necessarily a coalgebra morphism as well.

We know that neither $\beta = \tau$ nor, consequently, $\beta = \tau^{-1}$ fulfills this condition unless $\tau_{B,H} \circ \tau_{H,B} = \mathrm{id}_{H \otimes B}$. Although we have seen that there is no reasonable tensor product bialgebra $B \otimes H$ defined for bialgebras in $\mathscr{A}$, there is a perfectly natural notion of tensor product bialgebra if at least H is in the center of $\mathscr{A}$-Coalg. But note that for the same reasons as mentioned above, the morphism $\sigma_{H,B}$, which completes H to an object of $\mathscr{Z}(\mathscr{A}\text{-Coalg})$, is not a bialgebra morphism.

Theorem 10. *Let B be a bialgebra in $\mathscr{A}$, and H an algebra in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$. Then $B \otimes H$ is a bialgebra with comultiplication and multiplication defined by*



and



.

If also B is an algebra in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$, then this $B \otimes H$ is an algebra in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$, namely the tensor product algebra of B and H in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$.

If $\mathscr{C}$ and $\mathscr{D}$ are monoidal categories then $\mathscr{C} \times \mathscr{D}$ is also monoidal. If $\omega : \mathscr{C} \to \mathscr{A}$ and $v : \mathscr{D} \to \mathscr{A}$ are monoidal functors, then so is $\omega \otimes v$. Note that $(\mathscr{C} \times \mathscr{D}, \omega \otimes v)$ is the tensor product algebra of $(\mathscr{C}, \omega)$ and $(\mathscr{D}, v)$ in $\mathfrak{C}$. The functor $\mathrm{coend}_{\mathscr{Z}}$ preserves tensor products, and thus:

Theorem 11. *Let $(\mathscr{C}, \otimes, \omega)$ and $(\mathscr{D}, \otimes, v)$ be monoidal categories over $\mathscr{A}$. Then $\mathrm{coend}_{\mathscr{Z}}(\omega \otimes v) \cong \mathrm{coend}_{\mathscr{Z}}(\omega) \otimes \mathrm{coend}_{\mathscr{Z}}(v)$ as algebras in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$.*

Note that in a braided monoidal category $(\mathscr{C}, \tau)$ the tensor product of two commutative algebras A and B is commutative, iff $\tau_{B,A} \circ \tau_{A,B} = \mathrm{id}_{A \otimes B}$. So if ω and v are braided functors $\omega \otimes v$ will not be braided in general.

If $\mathscr{C}$ and $\mathscr{D}$ are rigid categories, then by [10] both $\mathrm{coend}(\omega)$ and $\mathrm{coend}(v)$ are Hopf algebras, and since $\mathscr{C} \times \mathscr{D}$ is also rigid, $\mathrm{coend}(\omega \otimes v)$ is also a Hopf algebra. Thus the result of the following theorem is not a surprise. It shows however, that the notion of an algebra in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$ works well.

Theorem 12. *Let C, D be Hopf algebras in $\mathscr{A}$, with (D, σ) an algebra in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$. Then $C \otimes D$ is a Hopf algebra with antipode*

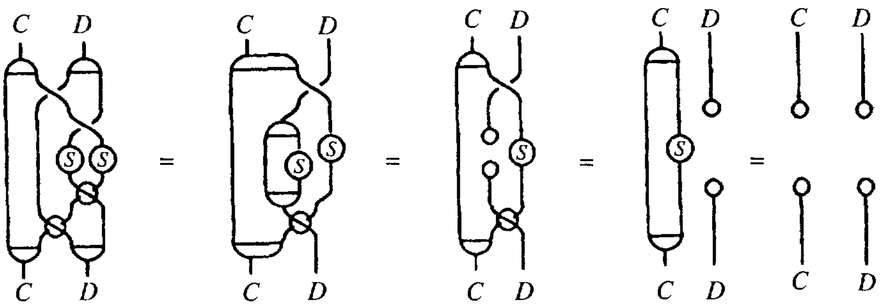
$$S_{C \otimes D} = \sigma_{v,C}(S_D \otimes S_C)\tau_{C,D}.$$

We can represent $S_{C \otimes D}$ in the pictorial calculus by

$S_{C\otimes D}$ =

.

Proof. We have to show $\nabla(1 \otimes S_{C\otimes D})\Delta = \eta\varepsilon$ and $\nabla(S_{C\otimes D} \otimes 1)\Delta = \eta\varepsilon$. We show the first equation, the second can be proved with similar reasoning.



To get the first equality we apply naturality of τ to Δ_D, and note that ∇_D is a morphism in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$. The second equality is the antipode property of S_D. The third step holds since τ is functorial (with respect to ε_D) and η_D is a morphism in $\mathscr{Z}(\mathscr{A}\text{-Coalg})$. The last step is the antipode property of S_C. $\square$

5. The transmuted Hopf algebra

In this section let k be a field and A be a k-Hopf algebra. Let $\langle\cdot \mid \cdot\rangle : A \otimes A \to k$ be a convolution invertible morphism such that $(A, \langle\cdot \mid \cdot\rangle)$ is a coquasitriangular (or braided) Hopf algebra in the sense of [8] or [5]. It is well known that the category $(\mathscr{M}^A, \otimes, k, \tau_{\langle\cdot \mid \cdot\rangle})$ of A-comodules is braided and monoidal. For any two A comodules $X, Y \in \mathscr{M}^A$, the braiding $\tau_{\langle\cdot \mid \cdot\rangle} : X \otimes Y \to Y \otimes X$ is defined by

$$X \otimes Y \ni x \otimes y \mapsto \sum y_{(0)} \otimes x_{(0)} \langle x_{(1)} \mid y_{(1)}\rangle \in Y \otimes X.$$

Since the forgetful functor $\mathscr{M}^A \to k\text{-Vec}$ creates colimits, $\mathscr{M}^A$ is cocomplete and $\otimes$ preserves arbitrary colimits in both variables. Thus all requirements for Tannaka duality are fulfilled. Let $\mathscr{M}^A_{\mathrm{f}}$ be the full subcategory of finite dimensional A-comodules. Then $\mathscr{M}^A_{\mathrm{f}}$, together with the forgetful functor $\mathscr{V} : \mathscr{M}^A_{\mathrm{f}} \to \mathscr{M}^A$ can be regarded as a category over $\mathscr{M}^A$. $\mathscr{M}^A_{\mathrm{f}}$ is monoidal, braided and rigid, hence coend($\mathscr{V}$) is a Hopf algebra in $\mathscr{M}^A$. We express this by saying coend($\mathscr{V}$) is an A-comodule Hopf algebra.

To apply Theorem 7, we need to know that our base category $\mathscr{M}^A$ is closed. The construction of an inner Hom functor for $\mathscr{M}^A$ is due to Ulbrich [9]. It is easy to check that for any $M, N \in \mathscr{M}^A$ the set $\mathrm{Hom}_k(M,N)$ is a left A^* module with $(h^* \cdot f)(m) = \sum f(m_{(0)})_{(0)} h^*(f(m_{(0)})_{(1)} S(m_{(1)}))$ for all $h^* \in A^*$, $m \in M$ and $f \in Hom_k(M,N)$. We define $\underline{\mathrm{hom}}(M,N)$ to be the unique maximal rational submodule and $\delta : \underline{\mathrm{hom}}(M,N) \to \underline{\mathrm{hom}}(M,N) \otimes H$ the corresponding comodule structure map. We assert that ($\underline{\mathrm{hom}}(M,N)$, δ) defines an inner Hom-functor. To prove this assertion, we only have to show that

$$\begin{aligned} \eta : \underline{\mathrm{hom}}(M,N) \otimes M &\to N \qquad \text{and} \quad \varepsilon : N \to \underline{\mathrm{hom}}(M, N \otimes N) \\ f \otimes x &\mapsto f(x) \qquad\qquad\qquad y \mapsto (x \mapsto y \otimes x) \end{aligned}$$

are A-comodule maps. This is easily done by direct calculations.

Let us review Majid's transmuted Hopf algebra construction, stated in [6]. Let $\mathscr{V} : \mathscr{M}_{\mathrm{f}}^A \to \mathscr{M}^A$ be the forgetful functor and define $(H, \Delta_H, \varepsilon_H, \mu_H, \eta_H, \lambda) := coend(\mathscr{V})$ as A-comodule Hopf algebra. We have then the following theorem:

Theorem 13. *As an A-comodule coalgebra H is isomorphic to* $(A, \Delta_A, \varepsilon_A; \delta^{\mathrm{ad}})$, *where*

$$\begin{aligned} \delta^{\mathrm{ad}} &: A \to A \otimes A \\ h &\mapsto \sum h_{(2)} \otimes S(h_{(1)})h_{(3)}. \end{aligned}$$

Let $a, b \in H$. *The multiplication* $\mu_H : H \otimes H \to H$ *is given by*

$$\mu(a \otimes b) = \sum a_{(2)}b_{(3)}\langle a_{(3)} \mid S(b_{(1)})\rangle\langle a_{(1)} \mid b_{(2)}\rangle$$

The antipode $\lambda : H \to H$ *is given by*

$$\lambda(a) = \sum S(a_{(2)})\langle S^2(a_{(3)})S(a_{(1)}) \mid a_{(4)}\rangle.$$

For all $(X, \delta_X^A) \in \mathscr{M}_{\mathrm{f}}^A$ *the H-comodule-structure* $\delta_X^{\mathscr{V}} : \mathscr{V}(X) \to \mathscr{V}(X) \otimes H$ *agrees with* $\delta_X^A : X \to X \otimes A$ *(in particular, this map is A-colinear).*

The functor $\mathscr{V} : \mathscr{M}_{\mathrm{f}}^A \to \mathscr{M}^A$ is braided. Therefore $(\mathrm{coend}(\mathscr{V}), \sigma_{\omega,-})$ is a commutative algebra in $\mathscr{Z}(A\text{-Coalg})$, where $\sigma_{\omega,-}$ is the natural isomorphism introduced in Proposition 5. Now we give a formula for σ and derive the explicit form of the commutativity relation $\mu = \mu \circ \sigma$.

Theorem 14. *Let V be any A-comodule. The morphism* $\sigma_{\omega,V}$ *from Proposition 5 reads*

$$\sigma_{\omega,V} : H \otimes V \to V \otimes H,$$

$$\sigma(a \otimes v) = \sum v_{(0)} \otimes a_{(2)}\langle v_{(1)} \mid a_{(1)}\rangle\langle a_{(3)} \mid v_{(2)}\rangle.$$

With this isomorphism, the coendomorphism Hopf algebra H obeys the following commutativity relation:

$$\mu = \mu \circ \sigma_{\omega,H} : H \otimes H \to H,$$

$$\mu(a \otimes b) = \sum \mu(b_{(3)} \otimes a_{(2)})\langle S(b_{(2)})b_{(4)} \mid a_{(1)}\rangle \; \langle a_{(3)} \mid S(b_{(1)})b_{(5)}\rangle.$$

Proof. The second claim follows from the first.

The morphism $\sigma_{\omega,V}$ is defined by: $(1 \otimes \sigma_{\omega,V})(\delta^{\omega} \otimes 1) = (\tau^2 \otimes 1)(1 \otimes \tau)(\delta^{\omega} \otimes 1)$. Now we apply the right side of this equation to $x \otimes v \in X \otimes V$.

$$\begin{aligned}
(\tau^2 \otimes 1) \circ (1 \otimes \tau) \circ (\delta_X \otimes 1)(x \otimes v) &= (\tau^2 \otimes 1) \circ (\textstyle\sum x_{(0)} \otimes \tau(x_{(1)} \otimes v)) \\
&= (\tau^2 \otimes 1)(\textstyle\sum x_{(0)} \otimes v_{(0)} \otimes x_{(2)} \langle S(x_{(1)}) x_{(3)} \mid v_{(1)} \rangle \\
&= (\tau \otimes 1)(\textstyle\sum v_{(0)} \otimes x_{(0)} \langle x_{(1)} \mid v_{(1)} \rangle \otimes x_{(3)} \langle S(x_{(2)}) x_{(4)} \mid v_{(2)} \rangle \\
&= \textstyle\sum x_{(0)} \otimes v_{(0)} \langle v_{(1)} \mid x_{(1)} \rangle \langle x_{(2)} \mid v_{(2)} \rangle \otimes x_{(4)} \langle S(x_{(3)}) x_{(5)} \mid v_{(3)} \rangle \\
&= \textstyle\sum x_{(0)} \otimes v_{(0)} \otimes x_{(4)} \langle v_{(1)} \mid x_{(1)} \rangle \; \langle x_{(2)} S(x_{(3)}) x_{(5)} \mid v_{(2)} \rangle \\
&= \textstyle\sum x_{(0)} \otimes v_{(0)} \otimes x_{(2)} \langle v_{(1)}) \mid x_{(1)} \rangle \langle x_{(3)} \mid v_{(2)} \rangle.
\end{aligned}$$

This proves the required equality, if we take into account that for any $h \in A$ we can find a finite dimensional subcomodule of A containing h. Now, if we specialize X to be this particular comodule, an application of $\varepsilon \otimes 1 \otimes 1$ yields the equation. □

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